

Hall-Littlewood symmetric functions via Yamanouchi toppling game

Robert Cori*, Pasquale Petrullo† and Domenico Senato‡

Abstract

We define a solitary game, the Yamanouchi toppling game, on any connected graph of n vertices. The game arises from the well-known chip-firing game when the usual relation of equivalence defined on the set of all configurations is replaced by a suitable partial order. The set of all firing sequences of length m that the player is allowed to perform in the Yamanouchi toppling game is shown to be in bijection with all standard Young tableaux whose shape is a partition of the integer m with at most $n-1$ parts. The set of all configurations that a player can obtain from a starting configuration is encoded in a suitable formal power series. When the graph is the simple path and each monomial of the series is replaced by a suitable Schur polynomial, we prove that such a series reduces to Hall-Littlewood symmetric polynomials. The same series provides a combinatorial description of orthogonal polynomials when the monomials are replaced by products of moments suitably modified.

Keywords: chip-firing game, Yamanouchi words, Young tableaux, Hall-Littlewood symmetric polynomials, orthogonal polynomials.

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*LaBRI Université Bordeaux 1, Email: cori@labri.u-bordeaux.fr

†Università degli Studi della Basilicata, Email: p.petrullo@gmail.com

‡Università degli Studi della Basilicata, Email: domenico.senato@unibas.it

1 Introduction

In [3] A. Björner, L. Lovász and P. Schor have studied a solitary game called the chip-firing game which is closely related to the sandpile model of Dhar [7]. In more recent papers some developments around this game were proposed. Musiker [18] introduced an unexpected relationship with elliptic curves, Norine and Baker [1] by means of an analogous game proposed a Riemann-Roch formula for graphs, for which Cori and Le Borgne [5] presented a purely combinatorial description. An algebraic presentation of the theory can be found in [2, 8, 19].

Given a graph \mathcal{G} with vertices v_1, v_2, \dots, v_n , one may consider any array $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of integers as a configuration associating to each vertex v_i the weight α_i . Suitable moves, here denoted by T_1, T_2, \dots, T_n and called topplings, can be performed in the game in order to change the starting configuration α into a new configuration β . Such moves can be reversed and this defines a relation of equivalence on \mathbb{Z}^n , here called toppling equivalence. The combinatorial interest of such a relation is grounded on its connections with several well-known combinatorial objects such as parking functions and Dick paths. In this paper we investigate more on this combinatorial game, which we refer to as the toppling game, by disclosing a wide range of connections with classical orthogonal polynomials and symmetric functions that we have outlined in [6].

Let $\alpha, \beta \in \mathbb{Z}^n$ and assume that β is obtained from α by successively performing topplings $T_{i_1}, T_{i_2}, \dots, T_{i_l}$. Then, we say that $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ is an α, β -toppling sequence and denote by $\mathcal{T}_{\alpha, \beta}$ the set of all such sequences. Given $(T_{i_1}, T_{i_2}, \dots, T_{i_l}) \in \mathcal{T}_{\alpha, \beta}$ then it is easily seen that $(T_{i_{w(1)}}, T_{i_{w(2)}}, \dots, T_{i_{w(l)}}) \in \mathcal{T}_{\alpha, \beta}$ for any permutation w of $1, 2, \dots, l$. Moreover, if $1 \leq k \leq l$ and if $\alpha^{(k)}$ is the configuration obtained from α via $(T_{i_1}, T_{i_2}, \dots, T_{i_k})$, then it is plain that $\alpha \equiv \alpha^{(k)}$, with \equiv denoting toppling equivalence. We will focus our attention on a restricted class $\mathcal{Y}_{\alpha, \beta} \subseteq \mathcal{T}_{\alpha, \beta}$ of toppling sequences that arise when toppling equivalence \equiv is replaced with a new relation \leq defined on \mathbb{Z}^n . A first crucial fact is that \leq is a partial order. Thus, instead of the whole classes of equivalent configurations, one may consider order ideals \mathcal{H}_α 's generated by all α 's. More concretely, one may also think \mathcal{H}_α as the set of all configurations β 's such that $\mathcal{Y}_{\alpha, \beta} \neq \emptyset$. In particular, one has $(T_{i_1}, T_{i_2}, \dots, T_{i_l}) \in \mathcal{Y}_{\alpha, \beta}$ if and only if the configuration $\alpha^{(k)}$, obtained from α via $(T_{i_1}, T_{i_2}, \dots, T_{i_k})$, satisfies $\alpha^{(k)} \leq \alpha$ for all $1 \leq k \leq l$. Therefore, an explicit characterization of $\mathcal{Y}_{\alpha, \beta}$ states that $(T_{i_1}, T_{i_2}, \dots, T_{i_l}) \in \mathcal{Y}_{\alpha, \beta}$ if and only if $i_1 i_2 \dots i_l$ is a suitable

Yamanouchi word over the alphabet of positive integers. This is why any sequence in $\mathcal{Y}_{\alpha,\beta}$ will be called a *Yamanouchi toppling sequence*.

Topplings also acts on the set $\{x^\alpha\}_{\alpha \in \mathbb{Z}^n}$, of monomials of the type $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, via $T_i \cdot x^\alpha = x^{T_i(\alpha)}$. In turn, this induces an action of the toppling group G (i.e. the group generated by T_1, T_2, \dots, T_n) on the ring of formal series $\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$. From this perspective, we may identify \mathcal{H}_α with the series

$$\mathcal{H}_\alpha(x) = \sum_{\beta \leq \alpha} x^\beta.$$

Since \leq is a partial order, then $\{\mathcal{H}_\alpha(x)\}_{\alpha \in \mathbb{Z}^n}$ is a basis of $\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$. By setting $T_{[i]} = T_1 T_2 \cdots T_i$ we can prove that the operator τ , defined by

$$\tau = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}},$$

satisfies

$$\tau \cdot x^\alpha = \mathcal{H}_\alpha(x),$$

for all $\alpha \in \mathbb{Z}^n$. A deformed version of τ , denoted $\hat{\tau}$, arises when elements of type $T_{[i,j]} = T_{[i]} T_{[i+1]} \cdots T_{[j-1]}$ are taken into consideration. More precisely, we set

$$\hat{\tau} = \prod_{1 \leq i < j \leq n} \frac{1}{1 - T_{[i,j]}},$$

and obtain a further basis $\{\hat{\mathcal{H}}_\alpha(x)\}_{\alpha \in \mathbb{Z}^n}$ satisfying

$$\hat{\mathcal{H}}_\alpha(x) = \hat{\tau} \cdot x^\alpha = \sum_{\beta \leq \alpha} C_{\alpha,\beta} x^\beta,$$

with $C_{\alpha,\beta}$ counting the number of pairwise distinct decompositions in terms of the generators $T_{[i,j]}$'s of the unique $g \in G$ such that $g(\alpha) = \beta$. At this point, one may introduce parameters z_1, z_2, z_3, q in order to keep track of the joint distribution of certain statistics $\ell_1, \ell_2, \ell_3, d$ defined on the set of all decompositions of any element in the toppling group. This is possible via a further deformation $\hat{\tau}(z_1, z_2, z_3, q)$ of $\hat{\tau}$, which leads to a parametrized version $\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x)$ of $\hat{\mathcal{H}}_\alpha(x)$. More precisely, we set

$$\hat{\tau}(z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - q) T_{[i,j]} z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}}}{1 - T_{[i,j]} z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}}},$$

and obtain

$$\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x) = \hat{\tau}(z_1, z_2, z_3, q) \cdot x^\alpha = \sum_{\beta \leq \alpha} C_{\alpha, \beta}(z_1, z_2, z_3, q; x) x^\beta.$$

Since $\hat{\tau}(z_1, z_2, z_3, q)^{-1} = \hat{\tau}\left(z_1, z_2, (1-q)z_3, \frac{q}{q-1}\right)$, then an explicit description of the series

$$\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x) = \hat{\tau}(z_1, z_2, z_3, q)^{-1} \cdot x^\alpha$$

is obtained for free,

$$\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x) = \sum_{\beta \leq \alpha} \left(\sum_{\ell_1, \ell_2, \ell_3, d} z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} (1-q)^{\ell_3-d} (-q)^d \right) x^\beta.$$

In the summation above, the values of $\ell_1, \ell_2, \ell_3, d$ range over all pairwise distinct factorizations $g = T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots$ of the unique $g \in G$ such that $g(\alpha) = \beta$. Since $\ell_1 \geq \ell_2 \geq \ell_3 \geq d \geq 0$ then the coefficient of x^β in $\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)$, as well as that in $\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x)$, is a polynomial with integer coefficients.

Noteworthy applications of this theory arise when Yamanouchi toppling is performed on the simple path with edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$. On one hand, the operator $\hat{\tau}$ reduces to a certain lowering operator arising within theory of symmetric functions and mapping Schur functions into Hall-Littlewood symmetric functions [9, 16]. As a consequence, the series $\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x)$ and $\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)$ reduce to Hall-Littlewood symmetric functions when α is a partition, when q, z_1, z_2, z_3 are suitably specialized, and when each x^β is suitably replaced by a Schur function. In turn, this enables us to define an analogue of Hall-Littlewood symmetric functions for any connected graph, thus opening the way to a systematic study of the matter. On the other hand, we can also prove that both $\hat{\mathcal{H}}_\alpha(q, z_1, z_2, z_3; x)$ and $\hat{\mathcal{K}}_\alpha(q, z_1, z_2, z_3; x)$ reduces to the $(n-1)$ th orthogonal polynomial $p_{n-1}(t)$, associated with a given linear functional with moments a_i 's, whenever $\alpha = (n-1, n-1, \dots, n-1, 0)$ and each x^β is replaced by $a_{\beta_1} a_{\beta_2} \cdots a_{\beta_{n-1}} t^{\beta_n}$. As an example, Hermite polynomials, Poisson-Charlier polynomials, Jacobi polynomials and any other classical orthogonal basis of the ring of polynomials in a single variable can be obtained by choosing the right sequence of moment (i.e. the right linear functional). Again, an analogue of classical orthogonal polynomials can be defined for any connected graph, with the chip-firing game concurring in giving a new combinatorial ground in common with symmetric functions.

2 Configurations on graphs, toppling game and Yamanouchi words

Here and in the following, by a graph \mathcal{G} we will always mean a connected graph $\mathcal{G} = (V, E)$, with set of vertices $V = \{v_1, v_2, \dots, v_n\}$, and with at most one edge $\{v_i, v_j\}$ for all $1 \leq i < j \leq n$. If the edge $\{v_i, v_j\}$ belongs to E then v_i and v_j will be said neighbors. A *configuration* on \mathcal{G} is a map, $\alpha: v_i \in V \mapsto \alpha(v_i) \in \mathbb{Z}$, associating to each vertex v_i an integral weight $\alpha(v_i)$. If we set $\alpha_i = \alpha(v_i)$ then we may identify any configuration α with the array $(\alpha_1, \alpha_2, \dots, \alpha_n)$. Henceforth, if $1 \leq i \leq n$, then $\epsilon_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$ will denote the configuration associating v_i with 1 and v_j with 0 if $j \neq i$. A *toppling* of the vertex v_i is a map $T_i: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ defined by

$$T_i(\alpha) = \alpha + \Delta_i, \quad (1)$$

where

$$\Delta_i = \left(\sum_{\{v_j, v_i\} \in E} \epsilon_j \right) - d_i \epsilon_i,$$

and $d_i = |\{v_j \mid \{v_i, v_j\} \in E\}|$ is the *degree* of v_i . Roughly speaking, the map T_i increases by 1 the weight α_j of each neighbor of v_i , and simultaneously decreases by d_i the weight α_i . As a consequence, the *size* $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ of any $\alpha \in \mathbb{Z}^n$ is preserved by each toppling T_i .

One may look at each T_i as a move of a suitable combinatorial game on the graph \mathcal{G} , that will be referred to as the toppling game. More precisely, assume that a starting configuration $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is given on \mathcal{G} , then label each vertex v_i with its own weight α_i . By “firing” the vertex v_i the starting configuration α is changed into a new configuration $\beta = T_i(\alpha)$. A *toppling sequence* on the graph \mathcal{G} simply is a finite sequence of fired vertices $(v_{i_1}, v_{i_2}, \dots, v_{i_l})$ or, equivalently, a finite sequence $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ of moves. We say that $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ is an α, β -*toppling sequence* to express that α can be changed into β by successively performing the corresponding moves, for short $\beta = T_{i_l} T_{i_{l-1}} \dots T_{i_1}(\alpha)$. It can be shown that a α, β -toppling sequence exists if and only if a β, α -toppling sequence exists. Then a relation of equivalence, called *toppling equivalence*, can be defined on \mathbb{Z}^n by setting $\alpha \equiv \beta$ if and only if an α, β -toppling sequence exists. Note that, the player of an α, β -toppling sequence $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ passes through intermediate configurations $\alpha = \alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(l)} = \beta$ defined by $\alpha^{(k)} = T_{i_k} T_{i_{k-1}} \dots T_{i_1}(\alpha)$

and satisfying $\alpha \equiv \alpha^{(k)}$. We are going to define an analogous game by replacing the toppling equivalence with a different relation on \mathbb{Z}^n .

From (1) one easily recover $T_i T_j = T_j T_i$ for all $1 \leq i, j \leq n$. This means that the set $\mathcal{T}_{\alpha, \beta}$ of all α, β -toppling sequences is closed under permutation of the topplings involved. In particular, this means that $\alpha \equiv \beta$ if and only if there exists $a \in \mathbb{Z}^n$ such that $T^a(\alpha) = \beta$, with $T^a = T_1^{a_1} T_2^{a_2} \dots T_n^{a_n}$ and $T_i^{a_i}(\alpha) = \alpha + a_i \Delta_i$.

Definition 1 (Toppling dominance). Let \mathcal{G} be a graph and let $\alpha, \beta \in \mathbb{Z}^n$. We say that α *dominates* β with respect to \mathcal{G} , written $\beta \leq \alpha$, if and only if

$$\beta = T^\lambda(\alpha) \text{ with } \lambda \in \mathbb{N}^n \text{ and } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

Now, assume $\beta \leq \alpha$ and assume that a player is asked to perform, if possible, an α, β -toppling sequence $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ which obeys the following prescription: for all $1 \leq k \leq l$, if $\alpha^{(k)} = T_{i_k} T_{i_{k-1}} \dots T_{i_1}(\alpha)$ then $\alpha^{(k)} \leq \alpha$. Henceforth, we will denote by $\mathcal{Y}_{\alpha, \beta}$ the set of all toppling sequences in $\mathcal{T}_{\alpha, \beta}$ that obey such a prescription.

Example 1. Let \mathcal{G} be the complete graph with vertices v_1, v_2, v_3, v_4, v_5 , then let $\alpha = (5, -3, 0, 1, -4)$ and $\beta = (-6, -4, 4, 5, 0)$. Straightforward computations show that both

$$(T_1, T_1, T_1, T_1, T_2, T_2, T_3, T_4, T_5) \text{ and } (T_5, T_4, T_3, T_2, T_2, T_1, T_1, T_1, T_1)$$

are α, β -toppling sequences. Nevertheless, the former sequence is in $\mathcal{Y}_{\alpha, \beta}$ instead of the latter which is in $\mathcal{T}_{\alpha, \beta} \setminus \mathcal{Y}_{\alpha, \beta}$. Moreover, the former sequence is not of minimal length since we also have $(T_1, T_1, T_1, T_2) \in \mathcal{Y}_{\alpha, \beta}$.

Hence, a first problem the player is going to face off is that of characterizing the set $\mathcal{Y}_{\alpha, \beta}$. A second matter is that of determining those sequences in $\mathcal{Y}_{\alpha, \beta}$ involving the minimum number of moves. One may easily realize that the α, β -toppling sequence $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ is in $\mathcal{Y}_{\alpha, \beta}$ if and only the following condition is satisfied: for all $1 \leq k \leq n$ and for all $2 \leq i \leq n$, the number of occurrences of T_i in $(T_{i_1}, T_{i_2}, \dots, T_{i_k})$ does not exceed the number of occurrences of T_{i-1} . So, if α is fixed and if we identify $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$ with the word $i_1 i_2 \dots i_l$, then $\bigcup_{\beta} \mathcal{Y}_{\alpha, \beta}$ exactly corresponds to the set of all Yamanouchi words over $\{1, 2, \dots, n\}$. Recall that, associated with each Yamanouchi word $w = i_1 i_2 \dots i_l$, and hence with each Yamanouchi toppling sequence $(T_{i_1}, T_{i_2}, \dots, T_{i_l})$, there is an integer partition $\lambda(w) = (\lambda_1, \lambda_2, \dots)$

whose i th part λ_i equals the number of occurrences of i in w . A suitable filling of the Young diagram of $\lambda(w)$ yields a coding of w in terms of a standard Young tableau. More precisely, the tableau associated with w is the unique tableau of shape $\lambda(w)$ whose i th row stores all j 's such that $i_j = i$. This provides a bijection between the set of all Yamanouchi words of l letters and the set of all standard Young tableaux of l boxes [21]. For instance, for the Yamanouchi word $w = 1\,1\,2\,1\,3\,2\,4$ we recover a standard Young tableau of shape $\lambda(w) = (3, 2, 1, 1)$,

$$w \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & 7 & \\ \hline \end{array}.$$

Note that, if the Yamanouchi words w and w' agree up to the order then $\lambda(w) = \lambda(w')$, so that the corresponding toppling sequences end at the same configuration β whenever their starting configuration is the same. However, the converse is not true. In fact, we have already noticed that, if \mathcal{G} is the complete graph with five vertices, the words $w = 1\,1\,1\,1\,2\,2\,3\,4\,5$ and $w' = 1\,1\,1\,2$, both change $\alpha = (5, -3, 0, 1, 4)$ into $\beta = (-6, -4, 4, 5, 0)$. However, their corresponding Young tableaux are of different size,

$$w \mapsto \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \hline 7 & & & \\ \hline 8 & & & \\ \hline 9 & & & \\ \hline \end{array} \quad \text{and} \quad w' \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}.$$

So, in order to get an explicit characterization of each set $\mathcal{Y}_{\alpha,\beta}$ we need a slightly deeper investigation.

3 The toppling group

Assume a graph \mathcal{G} is given and denote by T_1, T_2, \dots, T_n the corresponding toppling maps. The *toppling group* associated with \mathcal{G} is the group G generated by T_1, T_2, \dots, T_n . Since $T_i T_j = T_j T_i$ for all $1 \leq i < j \leq n$ then G is commutative. In particular, this says that all $g \in G$ may be expressed in terms of the T_i 's as $T^a = T_1^{a_1} T_2^{a_2} \dots T_n^{a_n}$, for a suitable array of integers $a = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. On the other hand, it is also easy to check that $T_1 T_2 \dots T_n(\alpha) = \alpha$ for all $\alpha \in \mathbb{Z}^n$ and for all \mathcal{G} . Thus, we have $T_1 T_2 \dots T_n = 1$, with 1 denoting the identity of G . As a consequence, if

$a - b = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ for some $k \in \mathbb{Z}$, then $T^a = T^b$. This means that each $g \in G$ admits a presentation T^a with $a \in \mathbb{N}^n$. In fact, if $g = T^b$ and if $b \notin \mathbb{N}^n$, then we set $k = \min\{b_1, b_2, \dots, b_n\}$ and choose $a = b - k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. Now, it is $a \in \mathbb{N}^n$ and $T^a = T^b(T_1 T_2 \dots T_n)^{-k} = g$. For instance, if $b = (-3, -1, 0, 2, 0, 0, 4, 0)$ then $k = -3$, $a = (0, 2, 3, 5, 3, 3, 7, 3)$ and finally

$$T^b = T^a = T_2^2 T_3^3 T_4^5 T_5^3 T_6^3 T_7^7 T_8^3.$$

In order to show that $T^a = T^b$ if and only if $a - b = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ with $k \in \mathbb{Z}$ we need a preliminary lemma.

Lemma 1. *We have*

$$T^a = 1 \text{ if and only if } a = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n).$$

Proof. If $a = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ then it is obvious that $T^a = 1$. Conversely, assume $T^a = 1$ and $a \in \mathbb{N}^n$. Let $k = \min\{a_1, a_2, \dots, a_n\}$ and set $b = a - k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. We obtain $T^a = T^b$ and $\min\{b_1, b_2, \dots, b_n\} = 0$. Assume $b_i = 0$ and consider T^b as a toppling sequence. Since $b_i = 0$, then v_i is not fired. Moreover, since the toppling sequence does not change the weight of v_i , then none of the neighbors of v_i have been fired. This means $b_j = 0$ whenever v_j is a neighbor of v_i . On the other hand, we may repeat the same reasoning for each neighbor of v_i and, since the graph is connected, in a finite number of steps we will have $b_i = 0$ for all $1 \leq i \leq n$. This exactly means $a = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. \square

Remark 1. Let $a, b \in \mathbb{Z}^n$ and assume $T^a(\alpha) = T^b(\alpha)$ for some α , equivalently, $T^{a-b}(\alpha) = \alpha$. If $c = (a - b) - k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ and if $k = \min\{a_1 - b_1, a_2 - b_2, \dots, a_n - b_n\}$, then $T^{a-b} = T^c$ and $\min\{c_1, c_2, \dots, c_n\} = 0$. Now, we may carry out a same reasoning as in the proof of the lemma above obtaining $T^c = 1$ and then $T^a = T^b$. That is $T^a(\alpha) = T^b(\alpha)$ if and only if $T^a = T^b$ and this means that if $\alpha \equiv \beta$ then there is a unique $g \in G$ such that $g(\alpha) = \beta$.

A first consequence of Lemma 1 is that the only relations satisfied by the generators of G are $T_i T_j = T_j T_i$ and $T_1 T_2 \dots T_n = 1$. This means that the group algebra $\mathbb{C}[G]$ of G is isomorphic to the ring

$$\frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle 1 - x_1 x_2 \dots x_n \rangle},$$

of polynomials $\mathbb{C}[x_1, x_2, \dots, x_n]$ modulo the ideal generated by $1 - x_1 x_2 \cdots x_n$. Moreover this provides an explicit characterization of all distinct presentations of any element in the toppling group G in terms of the generators T_1, T_2, \dots, T_n .

Theorem 2. *For all $a, b \in \mathbb{N}^n$ we have*

$$T^a = T^b \text{ if and only if } b - a = k(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n),$$

for a suitable $k \in \mathbb{Z}$.

Proof. Let $h = \max\{a_1, a_2, \dots, a_n\}$ and set $\tilde{a} = h(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) - a$. Clearly $T^a T^{\tilde{a}} = 1$ and, being $T^a = T^b$, also $T^b T^{\tilde{a}} = 1$. By virtue of Lemma 1 we deduce $b + \tilde{a} = j(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$ and then $b = a + k(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$ with $k = j - h$. \square

Now, once $g \in G$ and $a \in \mathbb{N}^n$ are chosen such that $g = T^a$, we may set $k = \min\{a_1, a_2, \dots, a_n\}$ and define $b = a - k(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$. Clearly $T^a = T^b$ and b is the unique element in \mathbb{N}^n of minimal size with this property. In other words, T^b is the unique *reduced decomposition* of $g = T^a$. Henceforth, we will denote by I_n the set of all $a \in \mathbb{N}^n$ satisfying $\min\{a_1, a_2, \dots, a_n\} = 0$. Moreover, we denote by P_n the set of all $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in I_n$ satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, hence $\lambda_n = 0$. Note that the map $a \in I_n \mapsto T^a \in G$ is a bijection. Furthermore, if zero entries are ignored then P_n can be identified with the set of all integer partitions with at most $n - 1$ parts. Finally, we can characterize each $\mathcal{Y}_{\alpha, \beta}$ in an explicit way.

Theorem 3. *If $\alpha, \beta \in \mathbb{Z}^n$ are such that $\beta \leq \alpha$ then there exists a unique $\lambda \in P_n$ such that*

$$T^\lambda(\alpha) = \beta,$$

and all sequences in $\mathcal{Y}_{\alpha, \beta}$ of minimal length are those associated with standard Young tableaux of shape λ

Proof. If $\beta \leq \alpha$ then there exists a unique $T^\eta \in G$ with $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n \geq 0$ and $\beta = T^\eta(\alpha)$. The reduced decomposition of T^η is given by T^λ , where $\lambda = \eta - \eta_n(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$. Hence, $(T_{i_1}, T_{i_2}, \dots, T_{i_l}) \in \mathcal{Y}_{\alpha, \beta}$ is an α, β -Yamanouchi toppling sequence of minimal length if and only if $i_1 i_2 \dots i_l$ is Yamanouchi of type λ , that is if and only if it associated with some standard Young tableaux of shape λ . \square

Corollary 4. *Let $\alpha, \beta \in \mathbb{Z}^n$ and assume $\beta = T^\lambda(\alpha)$ for some $\lambda \in P_n$. Then, all Yamanouchi toppling sequences associated with standard Young tableaux of shape μ are in $\mathcal{Y}_{\alpha, \beta}$ if and only if $\mu = \lambda + k(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$, with $k \in \mathbb{N}$.*

Proof. Let $i_1 i_2 \cdots i_l$ be Yamanouchi of type μ . We have $(T_{i_1}, T_{i_2}, \dots, T_{i_l}) \in \mathcal{Y}_{\alpha, \beta}$ if and only if $\beta = T^\mu(\alpha) = T^\lambda(\alpha)$. Then $T^\mu = T^\lambda$ and Theorem 2 assures us $\mu = \lambda + k(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$ for some $k \in \mathbb{N}$. \square

Example 2. Let \mathcal{G} denote the complete graph with five vertices, then assign $\alpha = (5, -3, 0, 1, -4)$ and $\beta = (-6, -4, 4, 5, 0)$. Since we have $\beta = T^\lambda(\alpha)$ for $\lambda = (3, 1, 0, 0, 0)$, then the minimum number of moves to pass from α to β is $4 = 3 + 1$. All α, β -Yamanouchi toppling sequences of minimal length are

$$(T_1, T_1, T_1, T_2), (T_1, T_1, T_2, T_1), (T_1, T_2, T_1, T_1).$$

They corresponds to the following standard Young tableaux,

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}$$

In the next section we will focus our attention on the set \mathcal{H}_α of all configurations that can be obtained from a given configuration α by means of any Yamanouchi toppling sequence.

Remark 2 (On the weight lattice of type A). In recent years, a general and beautiful algebraic theory of orthogonal polynomials have been developed in the framework of Hecke algebras associated with root systems [17]. For root systems of type A the associated orthogonal polynomials are the well-known Macdonald symmetric polynomials [16]. By comparing with Kirillov [10], one may check that for the Weyl group $W = A_{n-1}$ (i.e. the symmetric group \mathfrak{S}_n) the set I_n defined above can be identified with the weight lattice P . In turn, the set P_n agrees with the set P^+ of dominant weights. The subalgebra $\mathbb{C}[P]^W$ of the group algebra $\mathbb{C}[P]$ turns out to be isomorphic to the quotient

$$\frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle 1 - x_1 x_2 \cdots x_n \rangle}.$$

The generators of $\mathbb{C}[P]$ are usually denoted as formal exponentials e^λ 's, with $\lambda \in P$. This suggests the identification $T_i = e^{\epsilon_i}$. As we will show in the following sections, the toppling game provides an alternative and purely combinatorial way to recover common ground for symmetric and orthogonal polynomials. However, it remains the interesting question of a deeper understanding of possible connections between the toppling game and the whole theory developed in [17].

4 Generating series of configurations

For all $\alpha \in \mathbb{Z}^n$ we set

$$\mathcal{H}_\alpha = \{T^\lambda(\alpha) \mid \lambda \in P_n\} = \{\beta \mid \beta \leq \alpha\},$$

so that \mathcal{H}_α consists of all configurations that can be obtained from α by performing a Yamanouchi toppling sequence. Since the inverse of an element T^λ , with $\lambda \in P_n$, cannot be written in general as T^μ with $\mu \in P_n$, then toppling dominance is not a relation of equivalence.

Proposition 5. *Toppling dominance is a partial order on \mathbb{Z}^n .*

Proof. It is plain that \leq is a reflexive and transitive relation. Assume $\alpha \leq \beta$ and $\beta \leq \alpha$, so that $\beta = T^\lambda(\alpha)$ and $\alpha = T^\mu(\beta)$ with $\lambda, \mu \in P_n$. We deduce $\alpha = T^{\mu+\lambda}(\alpha)$ and so, via Lemma 1, $\lambda + \mu = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. By taking into account $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$, we may write $\lambda_i = k - \mu_i \geq k - \mu_{i+1} = \lambda_{i+1} \leq \lambda_i$, which forces $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ and also $\mu_1 = \mu_2 = \dots = \mu_n = 0$. We deduce $\alpha = \beta$, then \leq is antisymmetric. \square

In view of the proposition above, any set \mathcal{H}_α can be described as the principal order ideal generated by α .

Set $\mathbb{Z}[[G]] = \mathbb{Z}[[T_1, T_2, \dots, T_n]]$ and consider the following action of $\mathbb{Z}[[G]]$ on the ring $\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$ of all formal series in $x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}$. For all $a \in I_n$ and for all $\alpha \in \mathbb{Z}^n$ set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ and let

$$T^a \cdot x^\alpha = x^{T^a(\alpha)}.$$

By linear extension we obtain

$$\left(\sum_{a \in I_n} c_a T^a \right) \cdot \left(\sum_{\alpha \in \mathbb{Z}^n} d_\alpha x^\alpha \right) = \sum_{a \in \mathbb{N}^n} \sum_{\alpha \in \mathbb{Z}^n} c_a d_\alpha x^{T^a(\alpha)}.$$

Hence, any ideal \mathcal{H}_α uniquely determines the formal series

$$\mathcal{H}_\alpha(x) = \sum_{\beta \leq \alpha} x^\beta.$$

Consider the following element in $\mathbb{Z}[[G]]$,

$$\tau = \sum_{\lambda \in P_n} T^\lambda.$$

Since for all $\beta \in \mathcal{H}_\alpha$ there exists a unique $\lambda \in P_n$ such that $\beta = T^\lambda(\alpha)$, then we immediately recover

$$\tau \cdot x^\alpha = \sum_{\lambda \in P_n} T^\lambda \cdot x^\alpha = \sum_{\beta \leq \alpha} x^\beta = \mathcal{H}_\alpha(x).$$

Hence τ generates the whole order ideal \mathcal{H}_α by starting from the configuration α . Note that τ does not depend on α . Now, consider the following element in the toppling group,

$$T_{[i]} = T_1 T_2 \dots T_i \text{ for all } i = 1, 2, \dots, n-1.$$

Note that $T_{[i]}$ may be thought of as a Yamanouchi toppling sequence associated with a 1-column standard Young tableau.

Theorem 6. *Let $\lambda \in P_n$ and denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ the conjugate of λ . Then, T^λ has a unique expression in terms of the elements $T_{[i]}$, more precisely:*

$$T^\lambda = T_{[\lambda'_1]} T_{[\lambda'_2]} \cdots.$$

Proof. Let $\ell(\lambda)$ and $\ell(\lambda')$ denote the lengths of λ and λ' , respectively. Then we have

$$T^\lambda = \prod_{1 \leq i \leq \ell(\lambda)} T_i^{\lambda_i} = \prod_{1 \leq i \leq \ell(\lambda')} \left(\prod_{1 \leq j \leq \lambda'_i} T_j \right) = \prod_{1 \leq i \leq \ell(\lambda')} T_{[\lambda'_i]}.$$

□

Corollary 7. *We have*

$$\tau = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}}.$$

Proof. Consider the set $P'_n = \{\lambda' \mid \lambda \in P_n\}$. Clearly, P'_n is nothing but the set of all integer partitions whose largest part does not exceed $n-1$ and the map $\lambda \in P_n \mapsto \lambda' \in P'_n$ is a bijection. Then, via Theorem 6 we recover

$$\prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}} = \sum_{\mu \in P'_n} T_{[\mu_1]} T_{[\mu_2]} \cdots = \sum_{\lambda \in P_n} T^\lambda = \tau.$$

□

Now, we may write

$$\mathcal{H}_\alpha(x) = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}} \cdot x^\alpha.$$

Note that the action of each T_i on any x^α may be realized by suitably multiplying x^α by a monomial in $\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$. More precisely, we have

$$T_i \cdot x^\alpha = \left(x_i^{-d_i} \prod_{\{v_j, v_i\} \in E} x_j \right) x^\alpha.$$

This implies

$$T_{[i]} \cdot x^\alpha = \left(x_1^{-d_1} x_2^{-d_2} \cdots x_i^{-d_i} \prod_{k=1}^i \prod_{\{v_j, v_k\} \in E} x_j \right) x^\alpha,$$

so that we obtain

$$\frac{1}{1 - T_{[i]}} \cdot x^\alpha = \left(\frac{x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i}}{x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i} - \prod_{k=1}^i \prod_{\{v_j, v_k\} \in E} x_j} \right) x^\alpha, \quad (2)$$

and finally

$$\tau \cdot x^\alpha = \left(\prod_{i=1}^{n-1} \frac{x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i}}{x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i} - \prod_{k=1}^i \prod_{\{v_j, v_k\} \in E} x_j} \right) x^\alpha. \quad (3)$$

Identities (2) and (3) have to be intended in the following way: set $X_i = x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i}$ and $Y_i = \prod_{k=1}^i \prod_{\{v_j, v_k\} \in E} x_j$, then expand the right-hand side in (3) as a power series in Y_i/X_i , so that

$$T_{[i]} \cdot x^\alpha = \frac{X_i}{X_i - Y_i} x^\alpha = \sum_{n \geq 0} \left(\frac{Y_i}{X_i} \right)^n x^\alpha.$$

Example 3 (The complete graph $G = K_n$). We have

$$T_i \cdot x^\alpha = x^\alpha \frac{x_1 x_2 \cdots x_n}{x_i^n},$$

so that

$$T_{[i]} \cdot x^\alpha = x^\alpha \frac{(x_1 x_2 \cdots x_n)^i}{(x_1 x_2 \cdots x_i)^n},$$

and finally

$$\tau(x) = \prod_{i=1}^{n-1} \frac{(x_1 x_2 \cdots x_i)^n}{(x_1 x_2 \cdots x_i)^n - (x_1 x_2 \cdots x_n)^i}.$$

By expanding as a power series in $(x_1 x_2 \cdots x_n)^i / (x_1 x_2 \cdots x_i)^n$ we recover

$$\mathcal{H}_\alpha(x) = x^\alpha \prod_{i=1}^{n-1} \sum_{k \geq 0} \frac{(x_1 x_2 \cdots x_n)^{ki}}{(x_1 x_2 \cdots x_i)^{kn}}.$$

So we have the following theorem.

Theorem 8. *For each graph $\mathcal{G} = (V, E)$ there exists a formal power series $\tau(x) \in \mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$ such that*

$$\mathcal{H}_\alpha(x) = x^\alpha \tau(x),$$

for all $\alpha \in \mathbb{Z}^n$.

Elements $T_{[i]}$'s not only concur in giving an explicit expression for the operator τ , they also are algebraically independent and generate a subalgebra $\mathbb{C}[G]_{\geq}$ of $\mathbb{C}[G]$. More precisely, let $\mathbb{C}[G]_{\geq} = \mathbb{C}[T_{[1]}, T_{[2]}, \dots, T_{[n-1]}]$ and observe that, via Theorem 6, $\mathbb{C}[G]_{\geq}$ exactly is the subalgebra generated by all T^λ 's with $\lambda \in P_n$. A wider set of generators of $\mathbb{C}[G]_{\geq}$ is obtained by setting

$$T_{[i,j]} = T_{[i]} T_{[i+1]} \cdots T_{[j-1]} \text{ for all } 1 \leq i < j \leq n.$$

Note that each $T_{[i,j]}$ is a Yamanouchi toppling sequence associated with a tableaux whose conjugate shape consists of consecutive integers. Obviously $T_{[i]} = T_{[i,i+1]}$ so that the $T_{[i,j]}$'s generate the whole algebra $\mathbb{C}[G]_{\geq}$. On the other hand, the expression of each T^λ in terms of such generators is not unique. In order to find a *reduced decomposition* of $T^\lambda = T_{[\lambda'_1]} T_{[\lambda'_2]} \cdots$ in terms of the $T_{[i,j]}$'s we take the following path. Rearrange and associate the $T_{[\lambda'_i]}$'s so that

$$T^\lambda = (T_{[i_1]} \cdots T_{[j_1]})(T_{[i_2]} \cdots T_{[j_2]}) \cdots (T_{[i_k]} \cdots T_{[j_k]}),$$

and each of the sequences (i_h, \dots, j_h) 's consists of increasing consecutive integers. Then the reduced decomposition of T^λ is

$$T^\lambda = T_{[i_1, j_1+1]} T_{[i_2, j_2+1]} \cdots T_{[i_k, j_k+1]}.$$

Let us explicit the idea by means of a guiding example.

Example 4. If $\lambda = (8, 7, 4, 3, 2, 2, 1)$ then $\lambda' = (7, 6, 4, 3, 2, 2, 2, 1)$. Then we recover

$$\begin{aligned} T^\lambda &= T_{[7]}T_{[6]}T_{[4]}T_{[3]}T_{[2]}T_{[2]}T_{[2]}T_{[1]} = (T_{[1]}T_{[2]}T_{[3]}T_{[4]})(T_{[2]})(T_{[2]})(T_{[6]}T_{[7]}) \\ &= T_{[1,5]}T_{[2,3]}^2T_{[6,8]}. \end{aligned}$$

A reduced decomposition of T^λ is then $T_{[1,5]}T_{[2,3]}^2T_{[6,8]}$.

In general, any T^λ admits several reduced decompositions. For instance, both $T_{[1,3]}T_{[2,4]}$ and $T_{[1,4]}T_{[2,3]}$ are reduced decompositions of $T_1^4T_2^3T_3$. Hereafter, the total number of (reduced and non reduced) decompositions of T^λ will be denoted by $C(\lambda)$ or by $C_{\alpha,\beta}$ if $\alpha, \beta \in \mathbb{Z}^n$ and $\beta = T^\lambda(\alpha)$. A decomposition $T^\lambda = T_{[i_1,j_1]}T_{[i_2,j_2]} \cdots T_{[i_l,j_l]}$ is said to be *square free* if and only if each generator occurs with multiplicity at most one. For instance, $T_{[1,2]}T_{[1,3]}T_{[2,5]}$ is square free but $T_{[1,2]}^2T_{[2,3]}T_{[2,5]}$ is not. Also, note that $T_{[1,2]}T_{[1,3]}T_{[2,5]} = T_{[1,2]}^2T_{[2,3]}T_{[2,5]}$ so that any T^λ may have both square free and non square free decompositions. Now, consider the operator $\hat{\tau}$ defined by

$$\hat{\tau} = \prod_{1 \leq i < j \leq n} \frac{1}{1 - T_{[i,j]}}. \quad (4)$$

We recover

$$\hat{\tau} = \sum_{\lambda \in P_n} C(\lambda) T^\lambda,$$

so that we may write

$$\hat{\mathcal{H}}_\alpha(x) = \hat{\tau} \cdot x^\alpha = \sum_{\lambda \in P_n} C(\lambda) x^{T^\lambda(\alpha)} = \sum_{\beta \leq \alpha} C_{\alpha,\beta} x^\beta.$$

Roughly speaking the series $\hat{\mathcal{H}}_\alpha(x)$, as well as $\mathcal{H}_\alpha(x)$, is a generating series for the set \mathcal{H}_α . However, when $\hat{\tau}$ acts on the monomial x^α , each $\beta \in \mathcal{H}_\alpha$ is obtained $C_{\alpha,\beta}$ times.

At this point, any element in G , and in particular any T^λ , can be expressed in terms of three families of generators of the algebra $\mathbb{C}[G]_\geq$. Namely, we have the sets $\{T_i \mid i = 1, 2, \dots, n\}$, $\{T_{[i]} \mid i = 1, 2, \dots, n\}$ and $\{T_{[i,j]} \mid 1 \leq i < j \leq n\}$. Each family of generators gives rise to a notion of length of the decomposition, that is the total number of generators involved. More precisely, each T^λ admits a unique reduced decomposition in terms of the

T_i 's, whose length ℓ_1 equals the size of the partition $\lambda \in P_n$. Analogously, such a T^λ can be written in a unique way in terms of the $T_{[i]}$'s. In particular, we have $T^\lambda = T_{[\lambda'_1]} T_{[\lambda'_2]} \cdots$ and $\ell_2 = \lambda_1$ generators are involved. Moreover, each of the $C(\lambda)$ pairwise different decompositions of T^λ in terms of the $T_{[i,j]}$'s involves a certain number, say ℓ_3 , of generators. Finally, each reduced decomposition of T^λ in terms of the $T_{[i,j]}$'s involves a certain number, say d , of pairwise distinct generators. As a matter of fact, both ℓ_1 and ℓ_2 can be easily recovered once that a decomposition of T^λ in terms of the $T_{[i,j]}$'s is known. Indeed, assume we have

$$T^\lambda = T_{[i_1, j_1]}^{a_1} T_{[i_2, j_2]}^{a_2} \cdots T_{[i_d, j_d]}^{a_d},$$

with the $[i_k, j_k]$'s all distinct. Then, it is not difficult to see that¹

$$\ell_1 = \sum_{h=1}^d a_h \left(\binom{j_h}{2} - \binom{i_h}{2} \right),$$

and

$$\ell_2 = \sum_{h=1}^d a_h (j_h - i_h).$$

Example 5. Consider again $\lambda = (8, 7, 4, 3, 2, 2, 1)$ so that

$$T^\lambda = T_1^8 T_2^7 T_3^4 T_4^3 T_5^2 T_6^2 T_7.$$

We have $\ell_1 = |\lambda| = 8 + 7 + 4 + 3 + 2 + 2 + 1 = 27$. Moreover, being $\lambda' = (7, 6, 4, 3, 2, 2, 2, 1)$ then $T^\lambda = T_{[7]} T_{[6]} T_{[4]} T_{[3]} T_{[2]} T_{[2]} T_{[2]} T_{[1]}$ and in fact $\ell_2 = \lambda_1 = 8$. Now, consider the following decomposition of T^λ in terms of the $T_{[i,j]}$'s,

$$T^\lambda = T_{[1,5]} T_{[2,3]}^2 T_{[6,8]}.$$

It involves $\ell_3 = 4$ generators, and $d = 3$ among them are distinct. Finally, note that from $T^\lambda = T_{[1,5]} T_{[2,3]}^2 T_{[6,8]}$ we recover

$$\ell_1 = \binom{5}{2} - \binom{1}{2} + 2 \left(\binom{3}{2} - \binom{2}{2} \right) + \binom{8}{2} - \binom{6}{2} = 27,$$

and also

$$\ell_2 = (5 - 1) + 2(3 - 2) + (8 - 6) = 8.$$

¹where we assume $\binom{1}{2} = 0$

Theorem 9. *Set*

$$\hat{\tau}(z_1, z_2, z_3, q) = \sum_{\lambda \in P_n} \left(\sum_{\ell_1, \ell_2, \ell_3, d} z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} q^d \right) T^\lambda,$$

where the values of $\ell_1, \ell_2, \ell_3, d$ range over all pairwise distinct decompositions of T^λ in terms of the generators $T_{[i,j]}$'s. Then we have

$$\hat{\tau}(z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - q)T_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}}{1 - T_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}}. \quad (5)$$

Proof. We have

$$\begin{aligned} \hat{\tau}(z_1, z_2, z_3, q) &= \prod_{1 \leq i < j \leq n} \left(1 + \frac{qT_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}}{1 - T_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}} \right) \\ &= \prod_{1 \leq i < j \leq n} \sum_{k \geq 0} T_{[i,j]}^k q z_1^{k(\binom{j}{2} - \binom{i}{2})} z_2^{k(j-i)} z_3^k. \end{aligned}$$

Then, straightforward computations will give (5). \square

We may define a parametrized version of the series $\hat{\mathcal{H}}_\alpha(x)$ by setting

$$\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x) = \hat{\tau}(z_1, z_2, z_3, q) \cdot x^\alpha. \quad (6)$$

We recover

$$\hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x) = \sum_{\beta \leq \alpha} C_{\alpha, \beta}(z_1, z_2, z_3, q) x^\beta,$$

where the polynomial $C_{\alpha, \beta}(z_1, z_2, z_3, t)$ stores the values of $\ell_1, \ell_2, \ell_3, d$ relative to all pairwise distinct decompositions of the unique T^λ such that $\beta = T^\lambda(\alpha)$,

$$C_{\alpha, \beta}(z_1, z_2, z_3, q) = \sum_{\ell_1, \ell_2, \ell_3, d} z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} q^d.$$

In particular, we remark that for all $\beta \in \mathcal{H}_\alpha$ we have

$$C_{\alpha, \beta}(1, 1, 1, 1) = C_{\alpha, \beta}.$$

Moreover, we note that from (6) we gain a combinatorial description of the series $\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)$ defined by

$$\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x) = \hat{\tau}(z_1, z_2, z_3, q)^{-1} \cdot x^\alpha.$$

In fact, being $\hat{\tau}(z_1, z_2, z_3, q)^{-1} = \hat{\tau}\left(z_1, z_2, (1-q)z_3, \frac{q}{q-1}\right)$, we gain

$$\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x) = \hat{\mathcal{H}}_\alpha\left(z_1, z_2, (1-q)z_3, \frac{q}{q-1}; x\right)$$

so that the following combinatorial description is obtained,

$$\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x) = \sum_{\beta \leq \alpha} \left(\sum_{\ell_1, \ell_2, \ell_3, d} z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} (1-q)^{\ell_3-d} (-q)^d \right) x^\beta.$$

Since $\ell_1 \geq \ell_2 \geq \ell_3 \geq d \geq 0$ then also the coefficient $C'_{\alpha, \beta}(z_1, z_2, z_3, q)$ of x^β in $\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)$ is a polynomial with integer coefficients. It is related to $C_{\alpha, \beta}(z_1, z_2, z_3, q)$ by means of

$$C'_{\alpha, \beta}(z_1, z_2, z_3, q) = C_{\alpha, \beta}\left(z_1, z_2, (1-q)z_3, \frac{q}{q-1}\right).$$

By setting $z_1 = z_2 = z_3 = q = 1$ in $\hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)$ the only decompositions giving a nonzero contribution are those for which $\ell_3 - d = 0$, that is exactly square free decompositions.

5 Hall-Littlewood symmetric functions and Yamanouchi toppling

Let $x = \{x_1, x_2, \dots, x_n\}$ and denote by $\Lambda(x)$ the ring of symmetric polynomials in x with integer coefficients. For each positive integer i , let $h_i(x)$ denote the i th complete homogeneous symmetric polynomial, so that we have $\Lambda(x) = \mathbb{Z}[h_1, h_2, \dots]$ and then any $f(x) \in \Lambda(x)$ may be written in a unique way as a polynomial in the $h_i(x)$'s with integer coefficients. In particular, for all $\alpha \in \mathbb{N}^n$ we define

$$s_\alpha(x) = \det(h_{\alpha_i + j - i}(x))_{1 \leq i, j \leq n}.$$

The Jacobi-Trudi formula assures us that $s_\alpha(x)$ is a Schur polynomial if $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$, that is if α is an integer partition. Moreover, by swapping i th row and $(i+1)$ th row in the determinant above we have

$$s_\beta(x) = -s_\alpha(x) \text{ with } \beta = (\alpha_1, \dots, \alpha_{i+1} - 1, \alpha_i + 1, \dots, \alpha_n). \quad (7)$$

This is to say that any $s_\alpha(x)$ is zero or there is a partition λ such that $s_\alpha(x) = \pm s_\lambda(x)$. Now, consider the linear functional

$$E: x^\alpha \mapsto \begin{cases} s_\alpha(x), & \text{if } \alpha \in \mathbb{N}^n; \\ 0, & \text{if } \alpha \notin \mathbb{N}^n. \end{cases}$$

Hence, one may define symmetric polynomials by means of

$$E \hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x) \text{ and } E \hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x).$$

Next theorem states a first obvious but important fact.

Theorem 10. *For any graph \mathcal{G} , both*

$$\{E \hat{\mathcal{H}}_\alpha(z_1, z_2, z_3, q; x)\}_{\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0} \text{ and } \{E \hat{\mathcal{K}}_\alpha(z_1, z_2, z_3, q; x)\}_{\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0},$$

are bases of the ring $\Lambda(z_1, z_2, z_3, q; x)$ of symmetric polynomials in x with coefficients in $\mathbb{Z}[q, z_1, z_2, z_3]$.

The bases above gain particular interest in view of the special case when Yamanouchi toppling is performed on the simple path $\mathcal{G} = \mathcal{L}$ with edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}.$$

Lemma 11. *For the graph $\mathcal{G} = \mathcal{L}$ the generator $T_{[i,j]}$ realizes the lowering operator, more precisely for all $1 \leq i < j \leq n$ we have*

$$T_{[i,j]}(\alpha) = \alpha - \epsilon_i + \epsilon_j.$$

Proof. For this graph we have

$$T_i(\alpha) = \begin{cases} \alpha - \epsilon_1 + \epsilon_2, & \text{if } i = 1; \\ \alpha - 2\epsilon_i + \epsilon_{i-1} + \epsilon_{i+1}, & \text{if } 2 \leq i \leq n-1; \\ \alpha - \epsilon_n + \epsilon_{n-1}, & \text{if } i = n. \end{cases}$$

Then, it is not difficult to see that

$$T_{[i]}(\alpha) = T_1 T_2 \cdots T_i(\alpha) = \alpha - \epsilon_i + \epsilon_{i+1} \text{ for all } 1 \leq i \leq n-1,$$

so that

$$T_{[i,j]}(\alpha) = T_{[i]} T_{[i+1]} \cdots T_{[j-1]}(\alpha) = \alpha - \epsilon_i + \epsilon_j \text{ for all } 1 \leq i < j \leq n.$$

□

Let us recall that, if α is an integer partition then the Hall-Littlewood symmetric polynomial $R_\alpha(x; t)$ is defined by

$$R_\alpha(x; t) = \sum_{w \in \mathfrak{S}_n} w \left(x^\alpha \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j} \right).$$

Theorem 12. *Let $\mathcal{G} = \mathcal{L}$, then we have*

$$R_\alpha(x; t) = E \hat{\mathcal{K}}_\alpha(1, 1, t, 1; x) = \lim_{q \rightarrow 1} E \hat{\mathcal{H}}_\alpha \left(1, 1, (1-q)t, \frac{q}{q-1}; x \right).$$

Proof. For all $1 \leq i < j \leq n$ let $R_{j,i}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ denote the lowering operator defined by $R_{j,i}(\alpha) = \alpha - \epsilon_i + \epsilon_j$. From [16] we recover

$$R_\alpha(x; t) = \left\{ \prod_{1 \leq i < j \leq n} (1 - tR_{j,i}) \right\} \cdot s_\alpha(x), \quad (8)$$

where $R \cdot s_\alpha(x) = s_{R(\alpha)}$ for any product R of lowering operators, and $s_{R(\alpha)} = 0$ if $R(\alpha) \notin \mathbb{N}^n$. Observe that (8) can be rewritten as

$$R_\alpha(x; t) = E \left\{ \prod_{1 \leq i < j \leq n} (1 - tR_{j,i}) \right\} \cdot x^\alpha, \quad (9)$$

where we set $R \cdot x^\alpha = x^{R(\alpha)}$ for any product R of lowering operators. Now, set $\mathcal{G} = \mathcal{L}$ so that we have by the above Lemma $T_{[i,j]} = R_{j,i}$ for all $1 \leq i < j \leq n$. Then, (9) implies

$$\begin{aligned} R_\alpha(x; t) &= E \left\{ \prod_{1 \leq i < j \leq n} (1 - tT_{[i,j]}) \right\} \cdot x^\alpha \\ &= E \hat{\mathcal{K}}_\alpha(1, 1, t, 1; x) = \lim_{q \rightarrow 1} E \hat{\mathcal{H}}_\alpha \left(1, 1, (1-q)t, \frac{q}{q-1}; x \right). \end{aligned}$$

□

Remark 3 (Toppling dominance for \mathcal{L}). A further consequence of the fact that $T_{[i,j]}$ equals the lowering operator $R_{j,i}$ is that the restriction of toppling dominance to the set of all integer partitions of at most n parts reduces, when $\mathcal{G} = \mathcal{L}$, to the classical dominance ordering.

The coefficient of $s_\beta(x)$ in $R_\alpha(x; t)$ admits a description in terms of square free decompositions of elements in the toppling group of \mathcal{L} . More precisely, we recover

$$R_\alpha(x; t) = \sum_{\beta \leq \alpha} C'_{\alpha, \beta}(1, 1, t, 1) s_\beta(x).$$

Now, if $\lambda \in P_n$ is such that $T^\lambda(\alpha) = \beta$ then we can write

$$C'_{\alpha, \beta}(1, 1, t, 1) = \sum_{\ell_3} (-t)^{\ell_3},$$

where ℓ_3 ranges over the lengths of all square free decompositions of T^λ . This gives us the following formula for Hall-Littlewood symmetric polynomials.

Theorem 13. *Let α be an integer partition with at most n parts. Then, we have*

$$R_\alpha(x; t) = \sum_{\beta \leq \alpha} \left(\sum_{\ell_3} (-t)^{\ell_3} \right) s_\beta(x),$$

where ℓ_3 ranges over all lengths of all square free decompositions of the unique T^λ such that $T^\lambda(\alpha) = \beta$.

The most interesting transition matrix involving Hall-Littlewood symmetric polynomials arises by expanding Schur polynomials in terms of a normalized version of the $R_\alpha(x; t)$'s which is usually denoted $P_\alpha(x; t)$ [16]. The entries $K_{\lambda, \alpha}(t)$'s of this matrix, often called t -Kostka polynomials, are polynomials in t with positive integer coefficients. A celebrated combinatorial description, due to Lascoux and Schützenberger [15], expresses $K_{\lambda, \alpha}(t)$ as an enumeration of semistandard Young tableaux of shape λ and weight α with respect to the *charge* statistic. More recently, Haglund, Haiman, Loher and others have developed a beautiful combinatorial theory for Macdonald polynomials [11, 12, 13, 14]. This framework provides a new explanation of Lascoux-Schützenberger's result and extend the combinatorial description from Hall-Littlewood symmetric polynomials to Macdonald polynomials and their non-symmetric generalizations. The problem of finding a satisfactory

combinatorial description of t -Kostka polynomials, as well as that of finding an interpretation of Macdonald polynomials in terms of the toppling game, still remains open. When $t = 1$, $K_{\lambda,\alpha}(t)$ reduces to the Kostka number $K_{\lambda,\alpha}$. We close this section by giving an expression of Kostka numbers in terms of the coefficients $C_{\alpha,\beta}$'s.

Theorem 14. *Let $\mathcal{G} = \mathcal{L}$, let λ and μ be integer partitions with at most n parts, and assume λ dominates μ . Then, we have*

$$K_{\lambda,\mu} = \sum_w (-1)^w C_{\mu,w(\lambda)+w(\delta)-\delta},$$

with w ranging over the symmetric group on $1, 2, \dots, n$ and with $\delta = (n-1, n-2, \dots, 0)$.

Proof. The classical definition of the Schur polynomial states that

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}} = \det(x_i^{\lambda_j+i-j})_{1 \leq i,j \leq n} \prod_{i < j} \left(1 - \frac{x_j}{x_i}\right)^{-1}.$$

By expanding each factor $(1 - x_j/x_i)^{-1}$ as a formal power series in x_j/x_i the identity above still is true and, in particular, we obtain

$$\begin{aligned} s_\lambda(x) &= \prod_{1 \leq i < j \leq n} \frac{1}{1 - \frac{x_j}{x_i}} \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n} \\ &= \hat{\tau} \cdot \left(\sum_{w \in \mathfrak{S}_n} (-1)^w x^{w(\lambda)+w(\delta)-\delta} \right) \\ &= \sum_w \sum_{\beta \leq w(\lambda)+w(\delta)-\delta} (-1)^w C_{\beta,w(\lambda)+w(\delta)-\delta} x^\beta, \end{aligned}$$

where w ranges over all permutations of $1, 2, \dots, n$ and $\delta = (n-1, n-2, \dots, 0)$. On the other hand, it is well known that

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu} m_\mu(x).$$

By comparing the coefficient of x^μ we recover

$$K_{\lambda,\mu} = \sum_w (-1)^w C_{\mu,w(\lambda)+w(\delta)-\delta}.$$

□

6 Classical orthogonal polynomials and Yamanouchi toppling

Let us recall the notion of orthogonal polynomial system [4]. Assume that a linear functional $L: \mathbb{R}[t] \rightarrow \mathbb{R}$ is given. An *orthogonal polynomial system* associated with L is a polynomial sequence $\{p_n(t)\}_{n \in \mathbb{N}}$ such that $p_n(t) \in \mathbb{R}[t]$ and $\deg p_n = n$ for all $n \in \mathbb{N}$, and such that

$$L p_n(t) p_m(t) = 0 \text{ if and only if } n \neq m.$$

Let n be a positive integer and let \mathcal{L}_n denote the simple path with n vertices and with edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}.$$

Denote by $\hat{\tau}_n$ the operator $\hat{\tau}$ relative to \mathcal{L}_n , that is

$$\hat{\tau}_n = \prod_{1 \leq i < j \leq n} \frac{1}{1 - T_{[i,j]}}.$$

Moreover, set $\mathbf{x}_n = \{x_1, x_2, \dots, x_n\}$ and, for all $\alpha \in \mathbb{Z}^n$, define $q_\alpha(\mathbf{x}_n)$ to be the unique polynomial such that

$$\hat{\tau}_n \cdot q_\alpha(\mathbf{x}_n) = \mathbf{x}_n^\alpha, \tag{10}$$

where we set $\mathbf{x}_n^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Recall that we have

$$\hat{\tau}_n^{-1} = \prod_{1 \leq i < j \leq n} (1 - T_{[i,j]}) = \sum_{\lambda \in P_n} \left(\sum_{\ell_3} (-1)^{\ell_3} \right) T^\lambda,$$

where ℓ_3 ranges over all lengths of all square free decompositions of the fixed T^λ . Thus, we recover the following combinatorial formula for $q_\alpha(\mathbf{x}_n)$,

$$q_\alpha(\mathbf{x}_n) = \sum_{\beta \leq \alpha} \left(\sum_{\ell_3} (-1)^{\ell_3} \right) x^\beta.$$

Since the size of a configuration is preserved by any toppling sequence, then $q_\alpha(\mathbf{x}_n)$ is a homogeneous polynomial in x_1, x_2, \dots, x_n of total degree $|\alpha|$. To show how the polynomials $q_\alpha(\mathbf{x}_n)$'s are related to orthogonal polynomials

systems we need to manipulate polynomials with an arbitrary large number of variables at the same time. To this aim, we set $\mathbf{x} = \{x_1, x_2, \dots\}$ and define

$$\mathbb{R}[\mathbf{x}] = \bigcup_{n \geq 1} \mathbb{R}[\mathbf{x}_n].$$

Moreover, we will use maps $E: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ such that

1. for all $n \geq 1$ the restriction $E: \mathbb{R}[\mathbf{x}_n] \rightarrow \mathbb{R}$ is a linear functional;
2. for all $n \geq 2$, for all $p \in \mathbb{R}[\mathbf{x}_n]$ and for all $w \in \mathfrak{S}_n$,

$$E p(x_{w(1)}, x_{w(2)}, \dots, x_{w(n)}) = E p(x_1, x_2, \dots, x_n).$$

We will name E *symmetric functional*. Once that a symmetric functional E is given, for all $i \geq 1$ we may define a *conditional operator* $E_i: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}[x_i]$. Such an operator is uniquely determined by

$$E_i \mathbf{x}_n^\alpha = x_i^{\alpha_i} E \mathbf{x}_n^\alpha x_i^{-\alpha_i} \text{ for all } n \in \mathbb{N}, \text{ and for all } \alpha \in \mathbb{N}^n.$$

Roughly speaking, E_i acts on $\mathbb{R}[x_1, \dots, x_{i-1}, x_{i+1}, \dots]$ as E acts, and fixes each polynomial in $\mathbb{R}[x_i]$. We will say that the variables x_1, x_2, \dots are *independent* with respect to the functional E if and only if $E = E E_i$ for all $i \geq 1$, that is if and only if

$$E E_i p = E p, \text{ for all } p \in \mathbb{R}[\mathbf{x}] \text{ and for all } i \geq 1.$$

Note that the degree of $E_i q_\alpha(\mathbf{x}_n)$ does not exceed the maximum $k \in \mathbb{N}$ such that x_i^k occurs in $q_\alpha(x)$. Hence, we define $\{p_n(t)\}_{n \in \mathbb{N}}$ to be the unique polynomial sequence such that $p_0(t) = 1$, and such that

$$p_n(x_{n+1}) = E_{n+1} q_{(n,n,\dots,n,0)}(\mathbf{x}_{n+1}), \text{ for all } n \geq 1. \quad (11)$$

The following Theorem states an orthogonality relation for the polynomials defined in (11).

Theorem 15. *Let $E: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ be a symmetric functional, let x_1, x_2, \dots be independent with respect to E , and let $\{p_n(t)\}_{n \in \mathbb{N}}$ denote the unique polynomial sequence such that $p_0(t) = 1$ and satisfying (11). Then, for all $x_i \in \mathbf{x}$ we have*

$$E p_n(x_i) p_m(x_i) = 0 \text{ for all } n, m \in \mathbb{N} \text{ such that } n \neq m.$$

Moreover, if $\deg p_n = n$ for all $n \in \mathbb{N}$ then we also have

$$E p_n(x_i) p_n(x_i) \neq 0 \text{ for all } n \in \mathbb{N}.$$

Proof. Let $n, m \in \mathbb{N}$ with $0 \leq m < n$. Since $E = E E_{n+1}$ the, by comparing (10) and (11), we obtain

$$E x_{n+1}^m p_n(x_{n+1}) = E x_{n+1}^m x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \leq i < j \leq n+1} (x_i - x_j).$$

Let $w = (n+1, m+1) \in \mathfrak{S}_{n+1}$ denote the transposition exchanging $n+1$ and $m+1$. Since E is a symmetric functional then we have

$$\begin{aligned} E x_{n+1}^m x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \leq i < j \leq n+1} (x_i - x_j) \\ = E x_{w(n+1)}^m x_{w(2)} x_{w(3)}^2 \cdots x_{w(n)}^{n-1} \prod_{1 \leq i < j \leq n+1} (x_{w(i)} - x_{w(j)}). \end{aligned}$$

On the other hand, it is easily seen that

$$\begin{aligned} x_{w(n+1)}^m x_{w(2)} x_{w(3)}^2 \cdots x_{w(n)}^{n-1} \prod_{1 \leq i < j \leq n+1} (x_{w(i)} - x_{w(j)}) \\ = -x_{n+1}^m x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \leq i < j \leq n+1} (x_i - x_j). \end{aligned}$$

This forces $E x_{n+1}^m p_n(x_{n+1}) = -E x_{n+1}^m p_n(x_{n+1}) = 0$ for all $0 \leq m < n$. By linearity we recover,

$$E p_m(x_{n+1}) p_n(x_{n+1}) = 0 \text{ whenever } 0 \leq m < n.$$

Of course, the case $0 \leq n < m$ is analogous so that we conclude

$$E p_m(x_{n+1}) p_n(x_{n+1}) = 0 \text{ for all } n, m \in \mathbb{N} \text{ such that } n \neq m.$$

Besides, the symmetry of E assures us that we may replace x_{n+1} with any of the variables x_i 's.

Let $p_{n,n}$ denote the leading coefficient of $p_n(x_{n+1})$, so that we may write

$$E p_n(x_{n+1}) p_n(x_{n+1}) = p_{n,n} E x_{n+1}^n p_n(x_{n+1}) = p_{n,n} E E_{n+1} x_{n+1}^n q_{n,n,\dots,n,0}(x_{n+1}).$$

From $E = E E_{n+1}$ we obtain

$$E p_n(x_{n+1}) p_n(x_{n+1}) = p_{n,n} E x_2 x_3^2 \cdots x_n^{n-1} x_{n+1}^n \prod_{1 \leq i < j \leq n+1} (x_i - x_j),$$

and finally

$$E p_n(x_{n+1}) p_n(x_{n+1}) = p_{n,n} E q_{(n,n,\dots,n,n)}(\mathbf{x}_{n+1}).$$

Therefore, by comparing (10) and (11) it is easy to see that

$$p_{n,n} = E x_1^{n-1} x_2^{n-1} \cdots x_n^{n-1} \prod_{1 \leq i < j \leq n} (x_i - x_j) = E q_{(n-1,n-1,\dots,n-1,n-1)}(\mathbf{x}_n).$$

We obtain

$$E p_n(x_{n+1}) p_n(x_{n+1}) = p_{n,n} p_{n+1,n+1}.$$

so hence

$$E p_n(x_{n+1}) p_n(x_{n+1}) \neq 0 \text{ for all } n \in \mathbb{N},$$

whenever $p_{n,n} \neq 0$ for all $n \geq 1$. \square

Theorem 15 gives us an explicit way to build up an orthogonal polynomial system associated with any linear functional $L: \mathbb{R}[t] \rightarrow \mathbb{R}$, provided it exists. In fact, define $E: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ to be the unique symmetric functional such that

$$L t^k = E x_i^k \text{ for all } i, k \in \mathbb{N}, i \neq 0.$$

One can easily check that $E = E E_i$ for all $i \in \mathbb{N}$ with $i \neq 0$. Thus we may define the polynomial sequence $\{p_n(t)\}_{n \in \mathbb{N}}$ such that $p_0(t) = 1$ and satisfying (11). Theorem above assures us that, if $\deg p_n = n$ for all $n \in \mathbb{N}$, then we have

$$L p_n(t) p_n(t) = E p_n(x_{n+1}) p_n(x_{n+1}) = 0 \text{ if and only if } n \neq m,$$

and thus $\{p_n(t)\}_{n \in \mathbb{N}}$ is an orthogonal polynomial system associated with L . In turn, this means that the following combinatorial description of orthogonal polynomial systems can be given.

Theorem 16 (A combinatorial formula for orthogonal polynomials). *Assume that $\{p_n(t)\}_{n \in \mathbb{N}}$ is an orthogonal polynomial system with respect to some linear functional L , then we have*

$$p_n(t) = \sum_{\beta \leq (n,n,\dots,n,0)} \left(\sum_{\ell_3} (-1)^{\ell_3} \right) a_{\beta_1} a_{\beta_2} \cdots a_{\beta_n} t^{\beta_{n+1}},$$

where $a_i = L t^i$ denotes the i th oment of L , and where ℓ_3 ranges over all lengths of all square free decompositions of the unique $\lambda \in P_{n+1}$ such that $\beta = T^\lambda(n, n, \dots, n, 0)$.

By comparing this combinatorial formula with Theorem 13 one realizes the strong analogy between the expansion of an orthogonal polynomial in terms of the moments of the associated linear functional, and the expansion of a Hall-Littlewood symmetric polynomials in terms of the Schur functions. One might go a bit more into this analogy by considering more general families of graphs $\{\mathcal{G}_n\}_{n \geq 1}$, with \mathcal{G}_n having n vertices and \mathcal{G}_{n+1} obtained from \mathcal{G}_n by adding a vertex v_{n+1} and a certain number of edges. Hence, analogues of equations (10) and (11) can be given for a general \mathcal{G}_{n+1} , with $\hat{\tau}$ possibly replaced by $\hat{\tau}(z_1, z_2, z_3, q)$. Thus, a polynomial sequence $\{p_n(z_1, z_2, z_3, q; t)\}_{n \in \mathbb{N}}$ associated with any family $\{\mathcal{G}_n\}_{n \geq 1}$ is obtained. It reduces to classical orthogonal polynomial systems when $\mathcal{G}_n = \mathcal{L}_n$ and when $z_1 = z_2 = z_3 = q = 1$. This opens the way toward a general combinatorial theory for the analogues of the classical orthogonal polynomials, as well as of the classical symmetric functions, defined starting from a general family of graphs $\{\mathcal{G}_n\}_{n \geq 1}$.

Remark 4. The coding of orthogonal polynomials via symmetric functionals is at root of a deep connection among orthogonal polynomial systems and the invariant theory of binary forms. More on this subject, including a treatment of multivariable orthogonal polynomials, can be found in [20].

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